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Lyapunov-Based Controller for the Inverted Pendulum Cart System*

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Abstract. A nonlinear control force is presented to stabilize the under-actuated inverted pendulum mounted on a cart. The control strategy is based on partial feedback linearization, in a first stage, to linearize only the actuated coordinate of the inverted pendulum, and then, a suitable Lyapunov function is formed to obtain a stabilizing feedback controller. The obtained closed-loop system is locally asymptotically stable around its unstable equilibrium point. Additionally, it has a very large attraction domain.

Key words: Lyapunov-based control, nonlinear systems, under-actuated system

1. Introduction

Control of the simple inverted pendulum cart system has been one of the most interesting problems in modern control theory. The device consists of a pole whose pivot point is mounted on a cart. The pendulum is free to rotate about its pivot point. The cart can move horizontally perpendicular to the axis's pendulum and is actuated by a horizontal force. The control objective is to stabilize the mechanism around the unstable equilibrium point by applying a force to the cart. This system has attracted the attention of many researchers, as seen by a growing list of articles (for example, see [1–6]). The interest is due to the following facts. The system is non-feedback linearizable by means of dynamics state feedback [7], and hence, it is not linearizable by means of dynamic state feedback control either. The system loses controllability, when the pendulum swings past the horizontal. These obstacles makes it especially difficult to perform some controlled maneuvers; for instance, there is not a continuous force, which globally stabilizes the upright equilibrium of the pendulum with zero displacement of the cart [8]. However, the problem can be solved producing at least one discontinuity in the acceleration of the cart. Nevertheless, it is well known how to construct a linear locally stabilizing controller [9], but the linear-based control design has the inconvenience of having a very small domain of attraction.

In this article, we present a nonlinear control force to locally asymptotically stabilize the inverted pendulum cart system (IPCS), for a very large attraction domain, which is almost the whole upper half plane. The design of the proposed control law is based on Lyapunov's approach, in conjunction with a suitable partial feedback linearization of the IPCS. It is worth mentioning that partial feeback linearization of an under-actuated system has been considered in previous works. For instance in [10, 11], a methodology was proposed to control IPCS by swinging it up to its unstable equilibrium position, based on partial feedback linearization and Lyapunov technique. Also, Olfati-Saber in [2, 12] based

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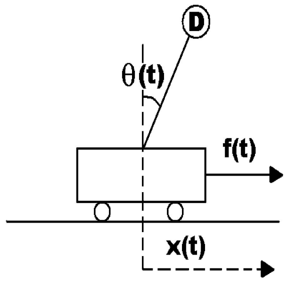

Figure 1. The IPC.

on a partial feedback linearization, developed a fixed point backstepping procedure for global and semi-global stabilization applicable for the stabilization of the IPCS over the upper half plane.

This paper is organized as follows. Section 1 presents the dynamic model of the IPCS and a partial feedback linearization of the nonlinear equations is proposed. Section 2 derives a nonlinear controller for the stabilization of the IPCS around its unstable equilibrium point. In the same section, we present some computer simulation results depicting the performance of the closed-loop system. Finally, Section 3 is devoted to presenting the conclusions.

2. Dynamic Model of the IPCS

Consider the traditional inverted pendulum mounted on a cart (see Figure 1). The nonlinear model of the system, which can be obtained from either the Newton or the Euler–Lagrage equations (for more detail see [13]) is given by

$$\begin{aligned} mL\cos\theta\ddot{x} + mL^2\ddot{\theta} - gmL\sin\theta = 0; \\ (M+m)\ddot{x} + Lm\cos\theta\ddot{\theta} - mL\dot{\theta}^2\sin\theta = f, \end{aligned} \tag{1}$$

where x is the cart displacement, θ the angle that the pendulum forms with the vertical, and f the force applied to the cart, acting as the control input. M and m stand for the cart mass and the pendulum mass concentrated in the bob, L is the length of the pendulum.

To simplify the algebraic manipulation in the forthcoming developments, we normalize the earlier equations by introducing the following scaling transformations,

$$q = x/L, \quad u = f/(mg), \quad d\tau = dt\sqrt{g/L}, \quad \delta = M/m \tag{2}$$

This normalization leads to the simple system,

$$\begin{aligned} \cos\theta\ddot{q} + \ddot{\theta} - \sin\theta = 0, \\ (1+\delta)\ddot{q} + \cos\theta\ddot{\theta} - \dot{\theta}^2\sin\theta = u, \end{aligned} \tag{3}$$

where, with an abuse of notation, "." stands for differentiation with respect to the dimensionless time τ. Then, a convenient partial feedback linearization input is proposed as follows (see [11]),

$$u = \cos\theta\sin\theta - \dot{\theta}^2\sin\theta + v(\sin^2\theta + \delta) \tag{4}$$

which produces the feedback equivalent system:

$$\ddot{\theta} = \sin\theta - \cos\theta v,$$
$$\ddot{q} = v. \tag{5}$$

Naturally, we may write the last differential equations as:

$$\dot{x} = f(x) + g(x)v, \tag{6}$$

with $x^T = (\theta, \dot{\theta}, q, \dot{q}) \in R^4$.[1]

We stress that for the new input $v = 0$ and $\theta \in [0, 2\pi]$ the aforementioned system has two equilibrium points; one is an unstable equilibrium point $x = 0$ and the other is a stable equilibrium point $x = (\pi, 0, 0, 0)$.

3. Lyapunov-Based Control Law

The main issue is to stabilize the system around its unstable equilibrium point, under assumption that the pendulum is initially above the horizontal plane. The problem will be solved by means of Lyapunov's method. Roughly speaking, the method consists of proposing a positive definite function (*or Lyapunov function*), provided that its time derivative along the trajectories of the system is, at least, semi-definite.[2] Finally, the asymptotic stability of the closed-loop system follows by applying LaSalle's invariance theorem.

3.1. How to Propose a Candidate Lyapunov Function

Let ξ_x be the following auxiliary variable defined as $\xi_x = q + k_p \sin\theta$, where k_p is a positive constant, and naturally, $\dot{\xi}_x = \dot{q} + k_p\dot{\theta}\cos\theta$.

The idea is to form a positive function V, provided that the time derivative along the trajectories of system (6) can be expressed as

$$\dot{V} = \dot{\xi}_x(\eta(x) + \delta(x)v) \tag{7}$$

where η and δ are some functions, with the restriction that $\delta(x) \neq 0$ for all $x \in D \subset R^4$.[3] Now, in order to built V, we propose:

$$V(x) = \frac{k_i}{2}\xi_x^2 + \frac{1}{2}\dot{\xi}_x^2 + k_l\phi(x), \tag{8}$$

where k_i and k_l are positive constants and, ϕ is a function selected such that

$$\frac{\partial}{\partial x}\phi(x)f(x) = 0, \quad \frac{\partial}{\partial x}\phi(x)g(x) = \dot{\xi}_x. \tag{9}$$

It turns out that $\phi(x)$ can be chosen as

$$\phi(x) = k_p(1 - \cos\theta) - \frac{k_p}{2}\dot{\theta}^2 + \frac{1}{2}\dot{q}^2. \tag{10}$$

[1] It can be easily seen that (6) defines an under-actuated system, because it has only one input v and two degrees of freedom θ and q.

[2] The very hard problem is how to find *the Lyapunov function*. Perhaps the most efficient technque used to built a Lyapunov's function is the proposed by Sepulchre et al. [14].

[3] The set D is a stability region for the closed-loop system, which will be determined later in the paper.

Moreover, it is easy to verify that the proposed V is positive definite for all $\theta \in I_S$, where the set I_S is defined as:

$$I_S = \{\theta \in R : |\theta| < \theta_S < \pi/2\}, \tag{11}$$

and $\theta_S = \cos^{-1}\sqrt{(1+k_l)/k_p}$.

Notice that the time derivative of V along the trajectories of system (6) can be expressed as:

$$\dot{V}(x) = \dot{\xi}_x(k_i\xi_x + \ddot{\xi}_x + k_l v) = \dot{\xi}_x(\eta(x) + \delta(x)v) \tag{12}$$

where

$$\begin{aligned} \eta(x) &= k_i\xi_x + k_p(\cos\theta - \dot{\theta}^2)\sin\theta, \\ \delta(\theta) &= 1 + k_l - k_p\cos^2\theta. \end{aligned} \tag{13}$$

3.2. A Nonlinear Feedback Control Law

Let us consider the aforementioned Lyapunov function V with its time derivative $\dot{V}$ given in Equations (8) and (12), respectively.

From Equation (12), a convenient control law is introduced as:

$$v = -\frac{\dot{\xi}_x + \eta(x)}{\delta(\theta)}, \tag{14}$$

which produces

$$\dot{V}(x) = -\dot{\xi}_x^2. \tag{15}$$

Notice that the control law (14) has no singularities, when angle $\theta \in I_S$. In order to avoid having $\delta(\theta) = 0$, it is sufficient that the initial conditions $x_0 = (\theta_0, \dot{\theta}_0, q_0, \dot{q}_0)$, with $|\theta_0| < \pi/2$ belonging to a neighborhood of the origin such that

$$V(x_0) < K_S = \frac{k_i k_p}{2}\sin\theta_S^2 + k_l k_p(1 - \cos\theta_S), \tag{16}$$

where θ_S was defined previously. Indeed, it follows since V is a non-increasing function, see relation (15).

It is important to emphasize that inequality (16) defines a stability region for the closed-loop system. Since, for all initial conditions x_0 such that $V(x_0) < K_S$, with $|\theta_0| < \pi/2$, then $V(x(t)) < K_S$, with $|\theta(t)| < \theta_S$. According to this fact, we can define a compact set Ω as:

$$\Omega = \{x = (\theta, \dot{\theta}, q, \dot{q}), |\theta| < \pi/2 : V(x) < K_S\}. \tag{17}$$

The set Ω has the property that all solutions of the closed-loop system (see Equations (5) and (14)) that start in Ω remain in Ω for ever. That is, Ω is an invariant set for the closed-loop system and it will be used to apply LaSalle's invariant Theorem [15].

3.3. Stability Analysis

Due to the fact that $V(x)$ is a positive definite function for all $x \in \Omega$, and $\dot{V}(x)$ is a semi-definite function for all $x \in R^4$, we conclude stability of the equilibrium point, in the sense of Lyapunov. To complete the proof, it is necessary to use LaSalle's invariance theorem.

Define the set:

$$S = \{x \in \Omega : (\dot{q} + k_p \dot{\theta} \cos\theta)^2 = \dot{\xi}^2 = 0\}. \tag{18}$$

Now, we need to compute the largest invariant set M in S.[4]

Evidently, on the set S, we have that $\xi = \bar{\xi}, \dot{\xi} = 0$, where $\bar{\xi}$ is a fixed constant. And from relations (12) and (14), it follows that the control law has been chosen as:

$$(k_i \xi_x + \dot{\xi}_x + k_l v) = (\eta(x) + \delta(x)v) = -\dot{\xi}, \tag{19}$$

and in the set S, it satisfies

$$k_i \bar{\xi} + k_l v = 0. \tag{20}$$

Then, on the set S, input v is evidently constant in S and given by the quantity $v = \bar{v}$. Now, from system (6), we have

$$\ddot{\theta} = \sin\theta - \cos\theta\,\bar{v}; \qquad \ddot{q} = \bar{v}. \tag{21}$$

We analyze the two possible cases arising from the earlier equations.

Case a: If the constant $\bar{v} \neq 0$ then $\dot{q}(t)$ is not bounded on S, and this fact leads to a contradiction. Hence, we must have that $\bar{v} = 0$. Additionally, when $\bar{v} = 0$ means that $\dot{q}(t)$ is constant on S, thus $q(t)$ is not bounded on S, and we also have a contradiction.

Case b: If $q = 0$ then from (21), we get the differential equation $\ddot{\theta} = \sin\theta$. This means that the pendulum can be either at rest or oscillating in S. Then, from definition of the variable ξ, it follows that $\theta = \sin^{-1}(\bar{\xi}/k_p)$, and necessarily, $\theta = 0$ or $\theta = n\pi$. But under the assumption that the initial conditions belong to Ω, we conclude that the variable θ must be zero.

We conclude that the largest invariant set M contained in the set S is constituted by the single point $x = 0$. According to LaSalle's theorem all the trajectories of the closed-loop system asymptotically converge towards the invariant set contained in S, which is the equilibrium point $x = 0$.

We summarize our results in the following proposition.

Proposition 1. *Consider the system* (6) *in the closed loop with the controller* (14), *with strictly positive constants k_i, k_l and k_p under assumption that the initial deviation of angle θ is in the upper half plane. Then, the origin of the closed loop system is locally asymptotically stable and the domain of attraction is the region defined by the inequality* (16).

[4] LaSalle's Theorem ensures that all solutions of the closed-loop system starting in $S \subset \Omega$ approach M as $t \to \infty$, where M is the largest invariant set in S.

3.4. A Brief Look at Stabilizing Control Laws for the IPCS

We will briefly describe the three most significant continuous controllers that render to zero the IPCS, assuming that the pendulum is initially above the horizontal plane. Maybe the most significant control law was presented by Bloch et al. [3, 16], based on the method of controlled Lagrangian in conjunction with the potential shaping energy. They derived an asymptotically and stabilizing control law, with a large domain of attraction. Other important approaches are based on the backstepping procedure introduced by Mazenc and Praly [17]. They first transform the original systems by suitable input to form two interconnected subsystems (this is carried out by means of nonlinear transformations), and then, by recursively applying the well-known backstepping method, they derive a nonlinear stabilizing controller that converges the angle position of the pendulum and the position of the cart to zero. The other similar stabilizing control law (see [2]), known as fixed point controller, consists in finding a cascade nonlinear system as a double integrator; then, by applying the backstepping procedure, fixed point equations for the control input are obtained and used to stabilize the aforementioned cascade system. There are many works related to the problem we are dealing with. Nevertheless, almost all of these works manage the problem by introducing some approximations, for instance, by reducing the order of the system or canceling some nonlinear dynamics. Besides, the closed-loop stability analysis of these approximations does not consider the computation of the domain of attraction, due to the fact that to obtain in a straightforward way a real estimation of this domain for a set of a forth-order system is, in general, a highly difficult problem (see [13]).

An exhaustive comparison among the previously introduced controllers and ours is beyond the scope of this paper. However, we consider that the closed-loop stability analysis of our control law turns out to be quite simple, compared to the three aforementioned controllers. This is because we use a Lyapunov function combined with LaSalle Theorem. Finally, our control law was inspired by the work of Lozano et al. [1]. The difference relies on the fact that we render the closed-loop solution to zero, while they conclude that the solution of the closed-loop system converges to an homoclinic orbit.

4. Numerical Simulations

To assess the performance of the proposed nonlinear control, we have carried out some digital computer simulations using the SIMNON simulation package.

In the first experiment, we used the proposed controller (14), when it was applied to the nonlinear model (6). The design parameters were set as $k_p = 25$, $k_i = 0.25$ and $k_l = 1$ and the initial conditions were set as $\theta_0 = 1.25$ rad, $\dot{\theta}_0 = -0.05$ rad/s, $q_0 = 0$ and $\dot{q}_0 = 0$ to show the large attraction domain. Figure 2 shows, in the normalized coordinates, the closed-loop performance of the IPCS to the proposed controller actions. Observe that even when the pendulum's initial angular position is far away from the top position and the angular velocity is downward, the closed-loop response still being quite effective. Intuitively, to counteract the effect of the pendulum's initial angular position and initial angular velocity, the cart has to make large horizontal displacements around the horizontal zero position (as it can be verified in Figure 2, where q is the horizontal cart displacement and $q \in [-20, 26]$).

In the second experiment, we considered the design parameters and the initial conditions as before, but introduced a dissipative force in the unactuated direction, i.e., we added the damping $-\gamma\dot{\theta}$ into the first differential equation of the model (6), with $\gamma = 0.25$. Figure 3 shows the robustness of the proposed nonlinear control when damping is considered in the numerical simulations. Notice that it is not generally true that damping makes a feedback-stabilized equilibrium asymptotically stable. That is to say, if damping is included then the response of the closed-loop system tends to destabilize.

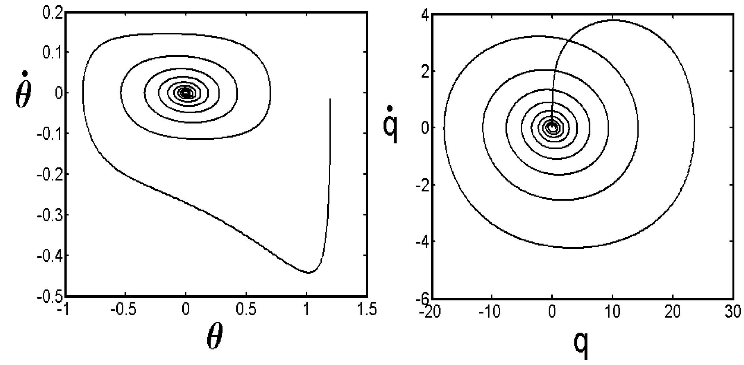

Figure 2. Feedback controlled performance of the normalized system to the proposed non-linear control.

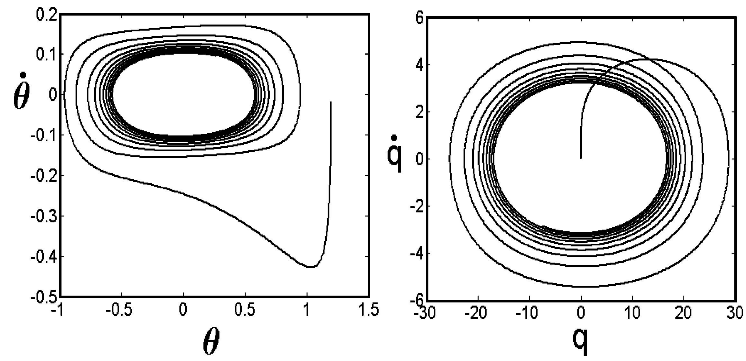

Figure 3. Closed-loop responses of Lyapunov-based controlled, when damping is included.

5. Conclusions

A Lyapunov-based controller for the stabilization of the IPCS was presented, in the case of the set of initial conditions belongs to the earlier horizontal plane. The proposed feedback controller makes the closed-loop system be locally asymptotically stable around the unstable equilibrium position $x = 0$. The domain of attraction of the closed-loop system is almost the whole upper half plane, and it can be estimated by inequality (16). The control strategy is based, in a first step, on partial feedback linearization of the IPCS, followed in a second step by Lyapunov's approach. The convergence analysis is carried out using LaSalle's invariance principle, which guarantees that the closed-loop system is locally asymptotically stable.

Finally, by means of computer simulations, the closed-loop performances of the controlled system seems to be quite robust with respect to the presence of dissipative forces, as assessed from the digital experiments. However, the proposed controller has the inconvenience of having a very slow time convergence.

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